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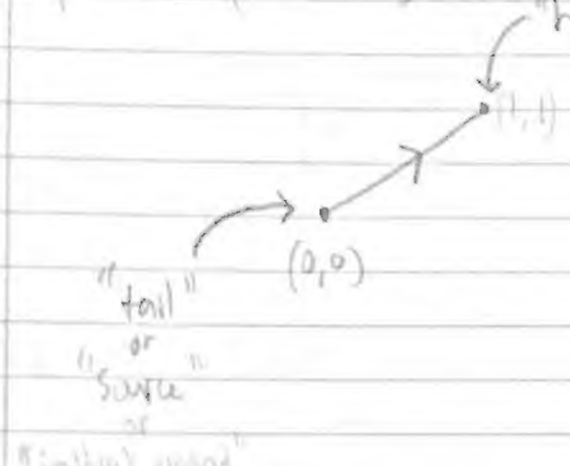
12.2 Vectors

Def: A vector in $\mathbb{R}^3$ is a directed line segment

picture (in $\mathbb{R}^2$)




Where two vectors are equivalent when they are "linear shifts" of one another

Operations on Vectors

1. Magnitude (Vector $\longmapsto$ real number $\geq 0$)

$|\vec{v}|$ = length of a segment representing $\vec{v}$


2. Addition (Vector + Vector $\longmapsto$ Vector)

$|\vec{v}|$ length $(\vec{st})$


"tip to tail"

$= \sqrt{(3-1)^2+(4-3)^2}$

$= \sqrt{4+1} = \sqrt{5}$

3. Subtraction (Vector − Vector $\longmapsto$ Vector)


$\vec{u}-\vec{v}$

"Tip to Tip"

Away from the tip you're subtracting

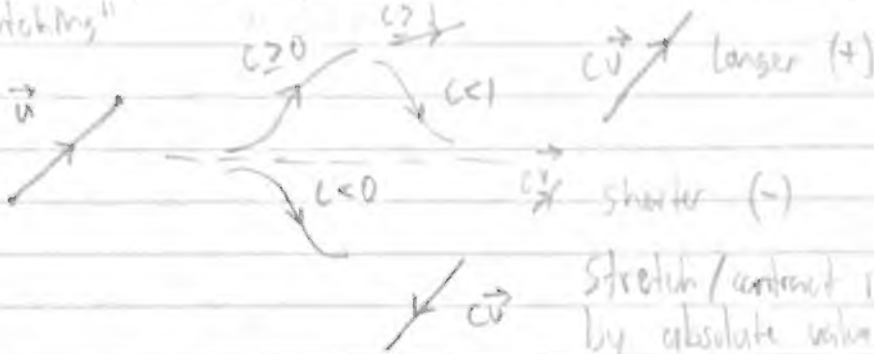
Ex: Zero Vector

The only vector with magnitude zero

4. Negation (Vector $\longmapsto$ Vector)

$-\vec{u}$ is obtained from $\vec{u}$ by "flipping" it


5. Scalar Multiplication ($\underbrace{\text{scalar}}_{}^{=\text{real number}} \cdot$ vector $\longmapsto$ vector)
"stretching"




Stretch/contract is determined by absolute value of c for negative

Every vector has a unique representation with tail at the origin

write $\vec{v} = \langle 1, 1 \rangle$ ← components of $\vec{v}$

To compute the component representation of $\vec{v}$, take a representation and compute "tip minus tail"

Ex: The vector represented by the segment situated from $(-3, 7)$ to $(-5, 11)$ has components

$$\vec{v} = \langle -5-(-3), 11-7 \rangle = \langle -2, 4 \rangle$$

Ex: The zero vector is the vector with all components 0.

in $\mathbb{R}^3$ : $\vec{0} = \langle 0, 0, 0 \rangle$
in $\mathbb{R}^4$ : $\vec{0} = \langle 0, 0, 0, 0 \rangle$

(Below we write in 3-space, but n-space is analogous)

① Magnitude

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{u}$ $\langle u_1, u_2, u_3 \rangle$ $\quad c \in$ Real #'s

$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $\leftarrow$ Immediate from the distance formula

② Addition

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

$\hookrightarrow$ componentwise addition!

③ Subtraction

$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$

$\hookrightarrow$ componentwise subtraction!

④ Negation

$-\vec{u} = \langle -u_1, -u_2, -u_3 \rangle$

⑤ Scalar Multiplication

$c\vec{u} = \langle cu_1, cu_2, cu_3 \rangle$

Adding two vectors only works if they belong to the same space

So $\langle 3, -7 \rangle + \langle 5, 1, 0 \rangle$ is nonsense!

Scalar Multiplication really needs scalar
$\vec{u}\,\vec{v}$ is nonsense!

Thus (Properties of Vector Operat[ions]

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $a, b \in$

① $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ Ass[ociative]

② $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ Commutativ[e]

③ $\vec{0} + \vec{v} = \vec{v}$ Identity

④ $\vec{v} + (-\vec{v}) = \vec{0}$ Negatives

⑤ $a(b\vec{v}) = (ab)\vec{v}$

⑥ $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

⑦ $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

Scala[r] regr[...]

⑧ $0\vec{v} = \vec{0}$ and $1\vec{v} = \vec{v}$

It's a really good exercise to prove the theorem (in $\mathbb{R}^3$

Direction

Prop: Given $\vec{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$|c\vec{u}| = |c||\vec{u}|$$

Also, $|\vec{u}| = 0$ if and only if $\vec{u} = \vec{0}$

Definition of direction: The direction of vector $\vec{v} \neq \vec{0}$ is the unit vector (i.e. vector of length 1) obtained from $\vec{v}$. That is, $\frac{1}{|\vec{v}|}\vec{v}$

$\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector

Why? $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = \left|\frac{1}{|\vec{v}|}\right||\vec{v}| = \frac{1}{|\vec{v}|}|\vec{v}| = 1$

Standard basis in $\mathbb{R}^3$:

$$\left.\begin{aligned}\vec{i} &= \langle 1,0,0\rangle \\ \vec{j} &= \langle 0,1,0\rangle \\ \vec{k} &= \langle 0,0,1\rangle\end{aligned}\right]$$

Form the standard basis for $\mathbb{R}^3$

Every vector $\vec{v} = \langle v_1, v_2, v_3\rangle$

$$= \langle v_1,0,0\rangle + \langle 0,v_2,0\rangle + \langle 0,0,v_3\rangle$$
$$v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle$$

$$= v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$